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LETTER TO THE EDITOR

The fractionisation phenomenon in the $(\lambda\phi^6)_{1+1}$ model

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Abstract. We analyse the fractionisation for a Dirac field coupled to a $(\lambda\phi^6)_{1+1}$ theory. Taking a simple Yukawa interaction we point out the no-fractionisation phenomenon in the background provided by the (anti) kink solution associated with the scalar field. If we choose a different coupling the half-integral fermionic number is possible and the analysis is particularly transparent using a previous bosonisation procedure.

It is a well known fact nowadays that in the presence of a non-trivial topological background, provided by the classical solution of scalar theories (kinks in systems with one spatial dimension and monopoles in the three-dimensional systems), the fermionic number associated with the vacuum state need not be an integer and can even be a transcendental function of the coupling constants of the theory [1]. This interesting phenomenon, usually referred to as fractionisation, can be studied according to many different schemes (diagrammatic techniques, anomalies, Levinson's theorem); however from a mathematical point of view it is related to the η invariant of the corresponding Dirac Hamiltonian [2]. This object, introduced in the analysis of the index theorem for non-compact manifolds, represents a conveniently regularised expression of the operator's spectral asymmetry.

We find a simple situation supposing that the interaction exhibits a charge conjugation symmetry C, which relates the states with energy $+|E|$ to those of energy $-|E|$. It happens that under these circumstances the physical interest of the problem, which in the most general scheme spreads over all the spectrum, will only be concentrated on the zero-energy eigenstates. In fact, if they are normalisable, each one of them adds minus one half to the vacuum fermionic number.

To detect the zero-energy solutions present in a model the general theorem to be applied is precisely the index theorem in open (non-compact) manifolds, as first stated by Callias, Boot and Seely [3]. In this scheme the topological essence of the phenomenon is transparent and shown already by observing that the index only depends on the values that the background scalar field adopts at infinity.

In any case, for systems with one spatial dimension the fractionisation phenomenon, in the presence of a charge conjugation symmetry C, has been related to the typical SUSY quantum mechanics situations. So the initial Dirac equation built over the kink $\phi_c(x)$, the superpotential in the new language, is easily transformed into a pair of Schrödinger-type equations including a supersymmetric content.

We remember that the conventional formulation of the SUSY quantum mechanics assumes the operators

$$S=\begin{bmatrix} 0 & Q^+ \\ Q & 0 \end{bmatrix} \qquad H=S^2 \tag{1}$$

where Q and Q^+ include the superpotential $W(x)$

$$Q=\left(-\frac{d}{dx}+W\right) \qquad Q^+=\left(\frac{d}{dx}+W\right) \tag{2}$$

and the supersymmetric flavour concentrated in the transformation

$$S\begin{bmatrix} u \\ v \end{bmatrix}=\sqrt{E}\begin{bmatrix} u \\ v \end{bmatrix}. \tag{3}$$

Going to the Schrödinger equations we have

$$\begin{bmatrix} H_- & 0 \\ 0 & H_+ \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}=E\begin{bmatrix} u \\ v \end{bmatrix} \tag{4}$$

which, in a more transparent form, is

$$\left(-\frac{d^2}{dx^2}+W^2+\frac{dW}{dx}\right)u=Eu \tag{5a}$$

$$\left(-\frac{d^2}{dx^2}+W^2-\frac{dW}{dx}\right)v=Ev. \tag{5b}$$

Moreover, the SUSYQM problems constitute an excellent laboratory for analysing the form in which sypersymmetry can appear spontaneously broken (we remember that SUSY remains unbroken only if the energy of the ground state is zero). Then Witten [4] introduced an order parameter $\Delta(\beta)$ which in certain cases can elucidate whether the SUSY breaking phenomenon has emerged. It can be outlined that this Witten index represents a measurement of the difference between bosonic and fermionic modes, all of zero energy (let us point out that for positive energies duplication occurs between both parts). Taking the regularisation used in [4], the order parameter adopts the form

$$\Delta(\beta)=\mathrm{Tr}[\exp(-\beta H_+)-\exp(-\beta H_-)]. \tag{6}$$

At first sight it may be thought that $\Delta(\beta)$ does not depend on the regularisation parameter β since only the zero-energy states contribute to the trace. However the explicit calculations carried out according to different approaches [5, 6] show that it is not only the Witten index which can exhibit β dependence, but sometimes such surprising final results as $\frac{1}{2}$ are obtained in the limit $\beta\to\infty$. Specifically, the value corresponds to the case in which the hypothetical ground state with zero energy exhibits a non-normalisable character. In any case the relation between the fractionisation phenomenon and the Witten index for the models with C invariance is transparent. If Δ finally settles on either 1 or -1 we can find a normalisable fermionic zero-energy solution, precisely the condition required in order to have a vacuum state with fermionic number $\frac{1}{2}$ or fractionisation phenomenon. In contrast, the final result of $\frac{1}{2}$ for Δ represents a non-normalisable fermionic zero-energy solution, inadequate to produce fractionisation. We summarise this curious situation in the following statement: an integer Δ leads to fractionisation while a value of $\frac{1}{2}$ for Δ thwarts the phenomenon.

We start by pointing out the simple form taken by $\Delta(\beta)$ in terms of the diagonal parts of the heat kernels associated with the H_- and H_+ Hamiltonians [5]

$$\Delta(\beta)=\int (K_+(x,x,\beta)-K_-(x,x,\beta))\,dx. \tag{7}$$

On the other hand the proper equations satisfied by the heat kernels are included in

$$\left(\frac{\partial}{\partial\beta}-\frac{\partial^2}{\partial x^2}\mp W^2+\frac{\mathrm{d}W}{\mathrm{d}x}\right)K_\pm=0. \tag{8}$$

Then for some simple situations it is possible to determine $K_\pm$ through (8) and after the integration procedure expressed in (7) the final $\Delta(\beta)$ value is found. However it is more useful to make a good use of the following expression, obtained in an appendix of [5]:

$$\frac{\mathrm{d}}{\mathrm{d}\beta}(K_+(x,x,\beta)-K_-(x,x,\beta))=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}+2W\right)K_+(x,x,\beta). \tag{9}$$

Assuming now that $\mathrm{d}W/\mathrm{d}x\to 0$ as $x\to\pm\infty$, we see that asymptotically $K_\pm$ is simply the conventional solution of the heat equation with a mass term corresponding to the value of W at infinity. In this way it is not a difficult task to obtain finally [5]

$$\frac{\mathrm{d}\Delta(\beta)}{\mathrm{d}\beta}=\frac{1}{\sqrt{4\pi\beta}}[W_+\exp(-\beta W_+^2)-W_-\exp(-\beta W_-^2)] \tag{10}$$

where W_+ and W_- represent the values of the superpotential at infinity. The last formula is the most useful expression of the Witten index if one wishes to analyse the fractionisation phenomenon.

Now we apply the scheme above to a $(\lambda\phi^6)_{1+1}$ theory including a Yukawa-type interaction between Dirac fermions, a model governed by the following Lagrangian density:

$$L=\tfrac{1}{2}(\partial_\mu\phi)^2-\tfrac{1}{2}\lambda^2\phi^2[\phi^2-(\mu/\lambda)]^2+\mathrm{i}\bar\psi\gamma^\mu\partial_\mu\psi-\mu\phi\bar\psi\psi \tag{11}$$

where μ and λ are positive constants with dimensions of mass. Taking only the scalar part we point out the existence of a kink solution [7]

$$\phi_\mathrm{k}(x)=[(\mu/2\lambda)(1+\tanh\mu x)]^{1/2} \tag{12}$$

which makes a smooth interpolation between the vacua $\phi=0$ and $\phi=(\mu/\lambda)^{1/2}$ (see figure 1). We can also obtain other solutions by setting $x\to -x$ and $\phi_\mathrm{k}\to-\phi_\mathrm{k}$. In particular we shall employ the antikink associated with (12)

$$\phi_\mathrm{ak}(x)=[(\mu/2\lambda)(1-\tanh\mu x)]^{1/2} \tag{13}$$

with an interpolation between $\phi=(\mu/\lambda)^{1/2}$ and $\phi=0$. Within this conventional scheme we study the fermionic problem over the background provided by the kinks associated

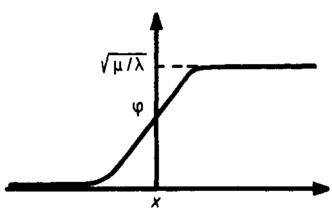

Figure 1. The kink ϕ^6 solution, interpolating smoothly between the vacua $\phi=0$ and $\phi=(\mu/\lambda)^{1/2}$.

with the purely bosonic part, without fermionic feedback to the ϕ field. Writing the spinor in its two-component form

$$\Psi(x, t) = \begin{bmatrix} u(x) \\ v(x) \end{bmatrix} \exp(\mathrm{i}\varepsilon t) \tag{14}$$

the general Dirac equation obtained from (11)

$$\left(\mathrm{i}\sigma_2 \frac{\mathrm{d}}{\mathrm{d}x} + \mu\phi\right) \Psi(x) = \varepsilon\Psi(x) \tag{15}$$

is reduced to the coupled pair

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + \mu\phi\right] v = \varepsilon u \tag{16a}$$

$$\left[-\frac{\mathrm{d}}{\mathrm{d}x} + \mu\phi\right] u = \varepsilon v \tag{16b}$$

which easily transforms into

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu\phi^2 + \mu\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) u(x) = \varepsilon^2 u(x) \tag{17a}$$

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu\phi^2 - \mu\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) v(x) = \varepsilon^2 v(x). \tag{17b}$$

If we choose for the scalar field the background provided by the kink solution (12) we recognise in (17) the conventional SUSY quantum mechanics exercise where the superpotential is simply

$$W(x) = [(\mu^3/2\lambda)(1 + \tanh \mu x)]^{1/2}. \tag{18}$$

Returning to (10), with W_- equal to zero according to the expression (18) for the superpotential, we find in this case

$$\frac{\mathrm{d}\Delta(\beta)}{\mathrm{d}\beta} = \frac{W_+}{\sqrt{4\pi\beta}} \exp(-\beta W_+^2). \tag{19}$$

If we now perform the integration with $\Delta(0) = 0$

$$\Delta(\beta) = \tfrac{1}{2}\Phi(W_+\sqrt{\beta}) \tag{20}$$

where Φ represents the probability Fresnel function. To recover a significant physical result the regularisation parameter β must tend to infinity [5] and so we find that $\Delta = \frac{1}{2}$, an identical value to the one obtained in the Liouville SUSY quantum mechanics theory [8]. We remember that in the bosonic sector of this model the zero-energy limit of the continuum-normalised wavefunction yields a non-square-integrable function which tends to zero at $x = \infty$, and to a constant at $x = -\infty$. Going to (3) in order to obtain the first-order equations for the zero modes we have

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + W\right] v = 0 \tag{21a}$$

$$\left[-\frac{\mathrm{d}}{\mathrm{d}x} + W\right] u = 0. \tag{21b}$$

Taking now the superpotential provided by the kink (18) we find that the hypothetical zero mode $v(x)$ behaves as

$$x \to \infty \qquad v(x) \to 0 \tag{22a}$$

$$x \to -\infty \qquad v(x) \to \left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)^{\sqrt{\mu}/2\sqrt{\lambda}} \tag{22b}$$

while the $u(x)$ diverges as $x \to \infty$. Choosing the antikink solution (13) the Witten index becomes $-\frac{1}{2}$ and then it is $u(x)$ which adopts the $v(x)$ behaviour written in (22). In either of the two cases, kink or antikink, there is no room for a fractionisation phenomenon with the Yukawa bosonic fermionic chosen in (11), since we cannot find a normalisable fermionic zero mode. Being a bidimensional theory, the $\bar{\Psi}\Psi U(\phi)$ interaction terms maintain the renormalisable character of the model, where $U(\phi)$ may be any well behaved function. Therefore if we introduce a coupling different from the Yukawa one the phenomenon would be recovered.

In particular the fractionisation is possible with an interaction term such as that in

$$L = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2}\lambda^2\phi^2[\phi^2 - (\mu/\lambda)]^2 + \mathrm{i}\,\bar{\Psi}\gamma^\mu\partial_\mu\Psi - \mu[3\phi^2 - (\mu/\lambda)]\bar{\Psi}\Psi. \tag{23}$$

Taking again the kink solution the superpotential is now

$$W(x) = (\mu^2/2\lambda)(3\tanh\mu x + 1). \tag{24}$$

Using (10) we have

$$\Delta(\beta) = \tfrac{1}{2}\Phi(W_+\sqrt{\beta}) + \tfrac{1}{2}\Phi(W_-\sqrt{\beta}) \tag{25}$$

with $W_+ = 2\mu^2/\lambda$ and $W_- = -\mu^2/\lambda$.

Again a Witten index has been obtained which is a non-trivial function of the regularisation parameter and the superpotential values at infinity; however when $\beta \to \infty$ we find that Δ takes the value 1, a logical result because from (21) it is evident that there exists a square-integrable fermionic state with zero energy, namely

$$\Psi_0(x) = v_0 \begin{bmatrix} 0 \\ (\cosh\mu x)^{-3\mu/2\lambda}\exp(-\mu^2 x/2\lambda) \end{bmatrix}. \tag{26}$$

The antikink solution leads to $\Delta = -1$ when β tends to infinity, with a non-zero upper component for the fermionic zero-energy state.

The fractionisation phenomenon admits another transparent treatment if we apply the bosonisation procedure [9]. In these circumstances choosing the general mass M, which depends on a precise normal-ordering prescription about the $\cos(2\pi^{1/2}\sigma)$ interaction, equal to the μ already present, the new bosonic potential of the model is

$$V(\phi, \sigma) = \tfrac{1}{2}\lambda^2\phi^2[\phi^2 - (\mu/\lambda)]^2 + \mu^2[3\phi^2 - (u/\lambda)]\cos 2\sqrt{\pi}\,\sigma. \tag{27}$$

Working within this bosonised version of the model an effective feedback of the fermionic part, now represented by σ, modifies the VEV for the ϕ field. In fact, looking for the minima of the $V(\phi, \sigma)$ potential, it is easy to find

$$\phi = 0 \qquad \sigma = q\sqrt{\pi} \tag{28a}$$

$$\phi = \pm a' \qquad \sigma = (m + \tfrac{1}{2})\sqrt{\pi} \tag{28b}$$

where q, $m \in z$ and

$$(a')^2 = (2 + \sqrt{19})\mu/3\lambda \tag{29}$$

which is obtained from the condition $\partial V/\partial\sigma = 0$. Resorting now to the Hessian matrix in order to guarantee the authentic minimal character, no conditions over the parameters of the theory are necessary.

If we remember the fermionic number assigned to a $\sigma(x)$ configuration [9]

$$Q_f = (1/\sqrt{\pi})(\sigma(\infty) - \sigma(-\infty)) \tag{30}$$

we can see that some kink-type solutions which interpolate between the minima of (28) exhibit half-integral fermionic numbers. To sum up, if we take kinks going from the $\phi = 0$ vacuum to the $\phi = a'$ vacuum then, the fractionisation phenomenon is recovered since the fermionic number supported by these solutions are necessarily non-integer, much the same as the result with the SUSYQM treatment.

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